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NON-LOCAL LIE PRIMITIVE SUBGROUPS OF LIE GROUPS

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ABSTRACT. Borovik found a Lie primitive subgroup of $E_8(\mathbb{C})$ isomorphic to $(\mathrm{Alt}_5 \times \mathrm{Sym}_6) : 2$. In this note, we provide a short proof of existence and his result that the conjugacy class of this subgroup is the only one among those of non-local Lie primitive subgroups of finite dimensional simple complex Lie groups having a socle with more than one simple factor.

1. **Introduction and statement of results.** In [CoGr 1987], the isomorphism types of finite nonabelian simple subgroups of the complex Lie groups $E_7(\mathbb{C})$ and $E_8(\mathbb{C})$ were studied. We define a *Lie primitive subgroup* of a complex Lie group to be a subgroup which is not contained in any proper, positive dimensional Zariski closed subgroup. In any group, a *local subgroup* is the normalizer of a nonidentity p-subgroup, for some prime number p. In [Aleks 1974] and, later, with different methods in [CLSS 1989], the local Lie primitive subgroups of complex simple Lie groups of exceptional type were classified.

Here, we continue the study of Lie primitive subgroups of a complex simple Lie group G of exceptional type. We show that any finite nonlocal Lie primitive subgroup of G normalizes a nonabelian simple subgroup, which, apart from a single exception found by Borovik, is unique up to conjugacy. Thus, we establish:

THEOREM 1.1. *Let G be an adjoint simple complex Lie group. Suppose L is a finite Lie primitive subgroup of G. Then either L is contained in a finite local subgroup or its socle is a nonabelian simple subgroup or $G = E_8(\mathbb{C})$ and $\mathrm{soc}\, L$ is isomorphic to* $\mathrm{Alt}_5 \times \mathrm{Alt}_6$. *Conversely there exists a subgroup of $E_8(\mathbb{C})$ isomorphic to* $\mathrm{Alt}_5 \times \mathrm{Alt}_6$ *which is Lie primitive and such a group is unique up to conjugacy.*

The above group of the form $\mathrm{Alt}_5 \times \mathrm{Alt}_6$ is called the *semisimple Borovik group* and its normalizer is called the *Borovik group*. The Borovik group contains the semisimple Borovik group with index 4 and it contains $\mathrm{Alt}_5 \times \mathrm{Sym}_6$ with index 2. More details on this group are given in § 4.

A more general version of this theorem (arbitrary characteristic of the ground field) has been announced by [Borov 1989] and, later, by [LiSe 1989]. We obtained these results independently and our treatment is relatively elementary and more detailed. The result (2.7), given by [LiSe 1989], considerably shortened an earlier version of this paper.

REMARK. The result is known for classical groups, for instance, by [Aschb 1984]. In fact, he points out a distinguished list of closed subgroups such that every finite group

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whose socle is not nonabelian simple is a subgroup of one of them. The members of that list are infinite except for normalizers of abelian subgroups, which come from nonabelian groups in the universal cover.

In a letter to one of us, Borovik exhibited a Lie primitive subgroup of $E_8(\mathbb{C})$ isomorphic to $(\mathrm{Alt}_5 \times \mathrm{Sym}_6) : 2$. Our construction of this group can be found in § 4.

We take this opportunity to report that the simple group Sz(8) with a ? should be on Table 2 of [CoGr 1987]. First of all, Sz(8) is in a 2-local subgroup of the sporadic group Ru. There is an embedding of Ru in $E_7(5)$ [KMR 1989]. Hence, by [Gr 1991, Appendix 2], the Borel subgroup of Sz(8), being of order prime to 5, lifts to $E_7(\mathbb{C})$ (an error is in (5.6.2) of [CoGr 1987]). From [GrRy 1992], we know that Sz(8) is contained in $E_7(K)$ for a field K if and only if char(K) is 2 or 5; the possibility that Sz(8) is embedded in $E_8(\mathbb{C})$ remains. Also, $U_3(8) : 12$ is now known to be embedded in $E_7(\mathbb{C})$ [GrRy 1992] and Ru is embedded in $E_7(5)$ [GrRy 1992] [KMR 1989]. An embedding of $L(2, 61)$ in $E_8(\mathbb{C})$ was proved recently [CoGrLi 1992]. Also, Lemma (3.5) of [CoGr] does not suffice to eliminate $L(4, 5)$, though its nonembedding in $E_8(\mathbb{C})$ follows trivially from the nonembedding of a $PSp(4, 5)$-subgroup. Finally, the second argument given to show the nonembedding of F_3 in $E_8(\mathbb{C})$ is not valid since the indicated element of order 3 need not have trace 5.

Another correction should be made to part (ii) of Theorem 1.1 of [CoGr 1987]; the groups SL(2, 31) and SL(3, 4) should be removed from the list, and the group $2 \cdot L(3, 4)$ should be inserted. The error is just a misstatement of our correct results (5.3.1) and (5.2.7) (which are correctly reported in Table 2).

A consequence of the above remarks, Theorem 1.1 of this article, [CoGr 1987] and [CoWa 1983, 1989] is that the isomorphism types of semisimple Lie primitive subgroups of exceptional Lie groups G are known, except for the few specific cases listed in [CoGr 1987] and [CoWa 1989].

2. **The setup.** Throughout this article, we shall denote by L a finite Lie primitive subgroup of G whose socle is denoted soc L and which is a direct product of finite nonabelian simple groups. Let N be a nonidentity normal subgroup of L such that $N \leq \mathrm{soc}\, L$. Then there exist $t \in \mathbb{N}$ and nonabelian simple subgroups N_i $(1 \leq i \leq t)$ such that $N = N_1 \times N_2 \times \cdots \times N_t$.

We assume that $t > 1$ and prove that N is the semisimple Borovik group; see (3.6) and § 4.

NOTATION 2.1. The adjoint module of G is denoted by $\mathbf{g}$, and the corresponding character of G by χ. By $E_7(\mathbb{C})$ we mean the adjoint group; its universal cover will be denoted by $2E_7(\mathbb{C})$. Similar notations for central extensions apply to the other simple Lie groups. By $1_a, 8_b, \ldots$ we mean an irreducible module of dimension 1, 8, *etc.* for some group or Lie algebra. The subscripts distinguish nonisomorphic modules of a given dimension. When the group is finite and essentially simple, we use the notation of [Atlas 1985]; otherwise, the symbols stand for well-known modules of the group, *e.g.*, $8_a, 8_b, 8_c$ stand for the complete set of 8-dimensional irreducibles for the Lie algebra or simply connected

Lie group of type D_4. The type of an element of finite order at most 7 in $E_8(\mathbb{C})$ is the label given to its conjugacy class in [CoGr 1987, Table 4].

Any irreducible representation of N is the tensor product of representations of the N_i $(1 \leq i \leq t)$. Thus, if $\psi_0^{(i)}, \ldots, \psi_{s_i}^{(i)}$ are the irreducible characters of N_i, the irreducible characters of N are of shape $\psi_{i_1}^{(1)} \otimes \psi_{i_2}^{(2)} \otimes \cdots \otimes \psi_{i_t}^{(t)}$, where $\otimes$ denotes character multiplication for a tensor product of modules for a direct product of groups. Hence there are non-negative numbers $a_{i_1,\ldots,i_t}$ such that

$$(*) \qquad \chi|_N = \sum_{i_1,i_2,\ldots,i_t} a_{i_1,\ldots,i_t} \psi_{i_1}^{(1)} \otimes \psi_{i_2}^{(2)} \otimes \cdots \otimes \psi_{i_t}^{(t)}.$$

In using this kind of decomposition, we will write the characters as in [Atlas 1985].

We recall

LEMMA 2.2 (*cf.* [CoGr 1987]). *A nontrivial normal subgroup of L has zero fixed point subalgebra on* $\mathbf{g}$.

PROOF. Let M be a nontrivial normal subgroup of L. The connected component C of the identity of the centralizer of M (for short: the *connected centralizer* of M) in G is normalized by the normalizer in G of M, whence by L. If M has nonzero fixed vectors in $\mathbf{g}$ then $C_{\mathbf{g}}(M)$ is a nontrivial subalgebra of $\mathbf{g}$; therefore $N_G(C)$ is a closed complex Lie subgroup of positive dimension containing L, contradicting Lie primitivity of L. ∎

We remark that, for N_i non-normal (so $t > 1$), (2.2) does not exclude $C_{\mathbf{g}}(N_i) \neq 0$, although eventually we shall see that this does not happen. Besides the connected centralizer of N, the lemma below gives another closed subgroup which is trivial.

LEMMA 2.3. *The subgroup* $\left(C_G\big(C_G(N_i)^{(\infty)}\big)^{(\infty)}\right)^{\circ}$ *is trivial, for all i.*

PROOF. Take distinct $i, j \in \{1, \ldots, t\}$. Clearly, $\prod_{k \neq i} N_k \leq C(N_i)^{(\infty)}$, so $1 < N_i \leq L_i := C\big(C(N_i)^{(\infty)}\big)^{(\infty)} \leq C(\prod_{k \neq i} N_k)^{(\infty)}$, which is proper in G since $t > 1$. Similarly, $L_j \leq C(\prod_{k \neq j} N_k)^{(\infty)} \leq C(N_i)^{(\infty)}$, whence $L_i \leq C(L_j)$. Thus, $\prod_k L_k$ is a proper algebraic subgroup of G normalized by L, so must be finite. Hence $L_k^{\circ} = 1$ for each $k \in \{1, \ldots, t\}$. ∎

COROLLARY 2.4. $\left(C_G\big(C_G(S)^{(\infty)}\big)^{(\infty)}\right)^{\circ} = 1$ *for any subgroup S of N_i, for each i.*

PROOF. Immediate from $C_G\big(C_G(S)^{(\infty)}\big)^{(\infty)} \leq C_G\big(C_G(N_i)^{(\infty)}\big)^{(\infty)}$ and the above lemma. ∎

LEMMA 2.5. *If, for each i, the group $C_G(N_i)$ has a solvable component group, the subgroup $C_G(C_G(N_i)^{\circ})$ is finite.*

PROOF. As in the previous lemma, it can be shown that the subgroups $C_G\big(C_G(N_i)^{\circ}\big)^{\circ}$ of G commute and that their product is normalized by L. ∎

LEMMA 2.6. *Let S be a finite simple subgroup of G. Then the component group of $C_G(S)$ is solvable or we are in an exceptional group and $C_G(S)$ is finite and nonsolvable.*

PROOF. Without loss, we may alter G by a convenient central extension or quotient. If G is of non-exceptional type, consider the standard representation on a complex vector space V, and the decomposition

$$V = \sum_{i \in I} V_i$$

of V into isotypical components V_i ($i \in I$). If G has type A_n then $C_G(S)$ is an algebraic group between a direct product of groups $\mathrm{GL}(V_i)$ and its commutator subgroup, whence the result. Suppose, next, that G is the commutator subgroup of the group stabilizing a nondegenerate alternating or symmetric bilinear form f. For $i \in I$, denote by i' the index in I for which V_i is contragredient to $V_{i'}$, and set $J = \{i \in I \mid |\{i, i'\}| = 2\}$. Then $C_G(S)$ is a subgroup containing the commutator subgroup of a direct product of the groups $\mathrm{GL}(V_i)$ (one for each pair $\{i, i'\} \in J$) and classical groups associated to the forms obtained by restricting f to the spaces V_i ($i \in I - J$), whence the result.

From now on, assume G is of exceptional type. Let S be a counterexample. Define $C := C_G(S)$. Then $R := C^{(\infty)} > C^\circ \cap C^{(\infty)}$ and C is infinite. Note that C is reductive (the centralizer of the reductive subgroup S) and that R is an algebraic group (equal to $C^{(k)}$, for sufficiently large k) and satisfies $C^{\circ\prime} \leq R$. Consequently, $R \cap C^\circ = Z_k C^{\circ\prime}$, for k sufficiently large, where $Z_k := R^{(k)} \cap Z(C^\circ)$. Since Z_k is an algebraic subgroup of a torus, it is reductive. Therefore, $R \cap C^\circ$ is reductive, whence so is R. Observe that if the reductive group $C^\circ_C(R)$ is not 1, it contains nontrivial semisimple elements outside $Z(G)$. We consider cases to obtain a contradiction.

CASE 1. $C_{C^\circ}(R)$ has a semisimple element $t \in C_{C^\circ}(R), t \notin Z(G)$. Then, $C_G(t)$ has solvable component group and has dimension less than that of G, so we finish by induction on the dimension upon passing to a quasisimple component Y of $C_G(t)^\circ$ such that $C_Y(S)/C_Y(S)^\circ$ is nonsolvable.

CASE 2. $C_{C^\circ}(R) = Z(G), C^\circ$ has quasisimple components and R has a nontrivial orbit on the set of quasisimple components. The components in this orbit must consist of groups H_i of type A_1 for $i \in J$, an index set of cardinality $n \geq 5$. Thus, G has type E_6, E_7 or E_8. Embed G in a group X of type E_8, altering G by a central extension or quotient if necessary. Since the 2-rank of G is at most 9 (by [Adams 1986], [CoSe 1987], [Gr 1991]), each H_i is isomorphic to $\mathrm{SL}(2, \mathbb{C})$. Let $H := \langle H_i \mid i \in J \rangle$ and let z_i be the central involution of H_i. Since R is perfect and $n \leq 8$, the action on the set of z_i is primitive. So, either the z_i are pairwise distinct or all equal.

We claim that the H_i are fundamental $\mathrm{SL}(2, \mathbb{C})$s in X.

CASE 2a. z_i has type $2A$. If $Z(H)$ contains a four group, V, of type AAA, H lies in $C(V) \cong T_2 E_6.2$, a contradiction to $V \leq H^{(\infty)}$. If $Z(H)$ contains a four group, V, of type AAB, H lies in a natural subgroup of type $A_1 A_1 D_6$. Without loss, we assume that there is no four group of type AAA in $Z(H)$. Embed a maximal torus of H in T, a maximal torus of X. With respect to the natural quadratic form on $\{x \in T \mid x^2 = 1\}$, $Z(H)$ is singular with respect to the bilinear form, but not the quadratic form, so has rank at most 4 and $Z(H) \cap 2A$ is the nontrivial coset of a codimension one subspace. On the other hand, it supports a group of automorphisms which is transitive on the n distinct z_i, so has rank at least 4, whence exactly 4. Therefore, from [CoGr], (3.8), we get that $C(Z(H)) \cong 2^4 A_1^8$. Since $|J| \geq 5$, at least one, hence all, of the H_i are fundamental $SL(2, \mathbb{C})$s. Finally, we suppose that the z_i are equal and seek a contradiction. Then, $H \leq C(z_i) \cong 2A_1E_7$. If H contains the A_1 factor, the factor must be normal in H and so must be one of the H_i, as required. So, we may assume that H does not contain the A_1 factor and so its image in the simple E_7 quotient is a direct product of n $PSL(2, \mathbb{C})$s. This implies that the 2-rank of adjoint E_7 is at least 10, in contradiction to [Gr 1991], (9.8.ii).

CASE 2b. z_i has type $2B$. If all z_i are distinct, then [CoGr 1987], (3.7) implies that H is in a group of type A_7 or D_4^2. If A_7, we get a contradiction by rank considerations. So, we may assume that $Z(H)$ contains no four group of type ABB. Thus, in any maximal torus T containing $Z(H)$, $Z(H)$ is a maximal isotropic subspace of $\{x \in T \mid x^2 = 1\}$ under the natural quadratic form. If D_4^2, we argue as in Case 2a to get H in a natural $2^4A_1^8$ and then verify the claim. Now assume that the z_i are all equal. We obtain a contradiction in this last case. Reindex to arrange $J = \{1, \ldots, n\}$. Let $P \cong \mathrm{Alt}_4$ be diagonally embedded in H_1H_2. If the involutions of $O_2(P)$ are of type $2B$, then $C_X(O_2(P)) \cong 2^2D_4^2 : 2$ and $H_3 \cdots H_nS$ is embedded in a product of at most two groups of type G_2 or A_2 (see [Tits 1959] or look ahead to (3.2)). Since these two groups have Lie rank at most two, at most two H_i project to a given factor and so, as $n \geq 5$, there is a pair i, j such that $H_i \cap H_j = 1$, a contradiction. If these involutions are of type $2A$, $H_3 \cdots H_nS$ is in Y, a natural E_6-subgroup. Since $H_3 \cdots H_nS$ contains $2^{1+2(n-2)} \times 2^2$, which has 2-rank $n - 1 + 2 \geq 6$, it follows from [Gr 1991] that if E is a subgroup of $H_3 \cdots H_nS$ of rank at least 6, it is toral of rank 6 and is maximal elementary abelian in Y and that $C_X(E)^{\circ\prime}$ is a natural $3A_2$-subgroup. Since H_1H_2 is not embeddable in $SL(3, \mathbb{C})$, we have a contradiction.

We now have that the H_i are fundamental $SL(2, \mathbb{C})$s. From [CoGr, 1987], (3.7), we know that the centralizer in X of two distinct such H_i has shape $2^2D_6.2$ and so the structure of D_6 implies that the connected centralizer of five such is a product of three fundamental $SL(2, \mathbb{C})$s (and lies in the subgroup $2^4A_1^8$ of [CoGr 1987], (3.8.i)). Since S is simple and $C(H)'$ contains the finite simple group S and is a *direct* product of at most three fundamental $SL(2, \mathbb{C})$s, we have a contradiction to the classification of finite subgroups of $SL(2, \mathbb{C})$.

CASE 3. $C_{C^\circ}(R) = Z(G)$, C° has quasisimple components and R has only trivial orbits on the set of quasisimple components. Thus, $R = C^{\circ\prime} \circ C_R(C^{\circ\prime})$, a central product. We get a contradiction by replacing G with a quasisimple component of $C_G(t)^\circ$, for some $t \in C^{\circ\prime} - Z(G)$ and using induction on the dimension; see the last remark in Case 1.

CASE 4. $C_{C^\circ}(R) = Z(G)$, C° has no quasisimple components, so is a torus. Set $T :=$ C°. Since C is infinite, $d := \dim(T) > 0$. By Case 1, $C_T(R) = Z(G)$. Let $D := C_G(T)$; if we embed T in a maximal torus T_0 and let Π be a root system, then D is generated by $N_G(T_0) \cap C_G(T)$ and those root groups centralizing T. Thus, D is connected and D' contains S, whence $\mathrm{rank}(D') \geq 1$. Also, the action of R on T corresponds to a subgroup of the Weyl group of G acting trivially on the subsystem Π' of roots associated to D'. Thus, R acts on T_0 as a subgroup of the Weyl group associated to Π'', the set of roots in Π perpendicular to those roots in Π'. Since R acts on T as a nontrivial perfect group, Π'' must have a connected component which contains an A_4 subsystem and $R/C_R(T)$ contains an element h of order 5. It follows that D' is generated by root groups in a natural simply connected subgroup $H = C_G([T_0, h])$ of type A_m, for some $m > 0, m \leq 4$.

In particular, D' is a nonempty direct product of at most two $\mathrm{SL}(n, \mathbb{C})$s and $Z(D') \cap Z(G) = 1$. Thus, R centralizes the nontrivial finite group $Z(D')$, which is in T but not in $Z(G)$, a contradiction. ∎

We owe part (i) of the following simple but powerful lemma to [LiSe 1989].

LEMMA 2.7. *Denote by n the product of all primes dividing the coefficients of the highest root when expressed as a linear combinations as fundamental roots. Thus $n = 30$ if $G = E_8(\mathbb{C})$ and $n = 6$ if $G = E_7(\mathbb{C})$, $E_6(\mathbb{C})$, $F_4(\mathbb{C})$ or $G_2(\mathbb{C})$. If G has type A_n, $n = 1$ and otherwise $n = 2$.*

(i) *If $x \in G$ is an element of finite order not equal to a coefficient of the highest root (in particular, if the order is prime to n), then the connected center $Z\big(C_G(x)\big)^\circ$ of $C_G(x)$ is nontrivial.*

(ii) *If X is a subgroup of G such that $C_G(X)^\circ = 1$, then each element $x \in C_G(X)$ satisfies $Z\big(C_G(x)\big)^\circ = 1$. In particular, $|x|$ divides 60.*

(iii) *For $E_8(\mathbb{C})$, the classes of finite order elements x such that $Z\big(C_G(x)\big)^\circ = 1$ are the following (below which are the component types of the centralizer):*

$1A$	$2A$	$2B$	$3A$	$3B$	$4A$	$4C$	$5C$	$6F$
E_8	A_1E_7	D_8	A_8	A_2E_6	A_7A_1	A_3D_5	A_4A_4	$A_5A_2A_1$.

PROOF. (i) Let $l = \mathrm{rank}(G)$ and let $(a_0, \ldots, a_l)$ be the labels of the extended Dynkin diagram ($a_0 = 1$ and the other a_i are coefficients of the highest root; see [Kac 1985], Chapter 4, Table Aff 1 for this and Chapter 8 for what follows). Elements of order m in $\mathrm{Inn}(G)$, up to conjugacy in $\mathrm{Aut}(G)$, are given, modulo diagram automorphisms, by assignments $(m_0, \ldots, m_l)$ of nonnegative integers to the nodes which generate the unit ideal of $\mathbb{Z}$ and satisfy $m = \sum_i a_i m_i$. Furthermore, the semisimple part of the centralizer of such an automorphism has as Dynkin diagram that subdiagram of the extended diagram which is supported at the set of those $i \in \{0, \ldots, l\}$ where $m_i = 0$. If x is an element of order m such that $Z(C_G(x))^\circ$ is trivial, this index set must have cardinality l, and if i is the unique index where m_i is nonzero, then (by the unit ideal condition) $m_i = 1$. Thus, $m = a_i$.

(ii) For $x \in C_G(X)$, we have $Z\big(C_G(x)\big) \leq C_G\big(C_G(x)\big) \leq C_G(X)$ whence $Z\big(C_G(x)\big)^\circ \leq C_G(X)^\circ = 1$.

(iii) Use (ii), [CoGr 1987] and the coefficients of the highest roots [Bour 1968]. ∎

COROLLARY 2.8. *$G = E_8(\mathbb{C})$, and, for all i, the order of N_i has no prime divisors greater than 5 and the centralizer of every element of N_i has trivial connected center. Furthermore, a Sylow 5-group of N_i has order 5 and there exists an involution of N_i inverting it under conjugation.*

PROOF. We first claim that every element of N_i has trivial connected centralizer. By Lemmas 2.5 and 2.6, $X := C_G(N_i)^\circ$ is trivial or has a finite centralizer. Suppose that $C_G(X)$ is finite. Then Lemma 2.7(ii) applies yielding that $Z\big(C_G(x)\big)$ is finite for each $x \in N_i$. According to Lemma 2.7(i), this implies that the order of N_i is as stated. Now assume that $X = 1$ and that the claim is false. There is an element $x \in N_i$ such that $Z := Z\big(C_G(x)\big)$ is nontrivial. Thus, for every index $j \neq i$, N_j centralizes Z and so $C_G(\langle N_j \mid j \neq i\rangle)$ is a positive dimensional closed subgroup. It is normalized by L (since $X = 1$) and we have a contradiction to Lie primitivity of L. The claim implies that $G = E_8(\mathbb{C})$ since the order of a nonabelian simple group requires at least three primes.

A Sylow 5-group has exponent 5, so it suffices to show that it does not contain a subgroup of the form 5×5. Suppose that A is such a group of order 25. Since A is a 2-generator finite abelian group, it is toral, so its centralizer has dimension at least 8. Orthogonality relations and the fact that traces of elements of order 5 here are all -2 (by (2.7.iii)) lead to a connected centralizer of dimension 8 exactly which therefore must be a torus, say T. Inspection of the centralizer of such an element of order 5 (shape $5A_4A_4$) shows that $C_G(A) \cong T : 5$, a solvable group. This is a contradiction since, for $j \neq i$, $N_j \leq C_G(A)$. (At this point, one could quote [Brauer 1968], which classifies finite simple groups of order 2^a3^b5 ($a, b \in \mathbb{N}$). The argument we choose in this article is more elementary.)

Burnside's famous normal p-complement theorem implies that, if P is a Sylow 5-group of N_i, there is $x \in N_{N_i}(P)$ which acts nontrivially on P. Since Aut(P) is cyclic of order 4 and $P = C_{N_i}(P)$, we may take x to be an involution. ■

LEMMA 2.9. *Suppose that $x_1, \dots, x_n$ are involutions from a torus of $G = E_8(\mathbb{C})$ and that each x_i is in 2A. Assume further, for each i, that S_i is a fundamental* SL(2, $\mathbb{C}$)*-subgroup containing x_i in its center (it is just the* SL(2, $\mathbb{C}$)*-factor in $C_G(x_i)$) and that, for each pair of indices $i \neq j$, $[S_i, S_j] = 1$. If the product $x_1 \cdots x_n$ is an involution, it is in 2B iff n is even.*

PROOF. Use the interpretation of involutions in the torus T as isotropic or anisotropic vectors in the vector space $\{x \in T \mid x^2 = 1\}$, according to whether they are in class $2B$ or $2A$. Under the natural bilinear form, two anisotropic vectors are orthogonal iff $[S_i, S_j] = 1$. Our hypotheses imply that the x_i generate a subspace of $\{x \in T \mid x^2 = 1\}$ which is totally singular with respect to the bilinear form. The products of evenly many x_i form a subgroup of index 2 consisting of the identity and the singular vectors. ■

COROLLARY 2.10. *For each i, N_i contains no element of order 6 and, for some i, N_i contains a subgroup isomorphic to* Alt_4.

PROOF. Suppose that N_i contains an element x of order 6. Then, x, x^2 and x^3 are in $6F$, $3B$ and $2A$, respectively. Let $j \neq i$ and let $D := \langle h, u\rangle$ be a subgroup of N_j which is dihedral of order 10, with $|h| = 5$ and $|u| = 2$; by (2.8), it is available. The centralizer of h has shape $5A_4A_4$ and u induces on each factor an outer automorphism whose fixed points form a copy of $SO(5, \mathbb{C})$. Let F_1 and F_2 be the two factors of type $5A_4$. For each index $l \neq j$. Each F_k meets N_l trivially, or else simplicity of N_l implies that $N_l \leq F_k$ and that a subgroup of order 5 in N_l meets $F_{k'}$ ($\{k, k'\} = \{1, 2\}$) trivially, against (2.7.iii). Thus, each N_l injects into each $F_k/Z(F_k)$ under the natural maps. By considering the natural 5-dimensional module for F_k, which contains $\langle N_l \mid l \neq j \rangle$, we conclude that $t = 2$. Suppose that N_i is normal in L. Since $C_G(x) \cong 6A_1A_2A_5$, (2.2) implies that N_j projects nontrivially to each factor, whence the classficiation of finite subgroups of $SL(2, \mathbb{C})$ implies that $N_j \cong \mathrm{Alt}_5$. But then, its image in the $6A_5$-factor is a reducible subgroup of the group $6A_5$ in its action on a 6-dimensional irreducible module and so $C(N_i)^\circ \neq 1$, against (2.2). We conclude that N_1 and N_2 are conjugate in L and so both contain elements of order 6. Thus, N_j centralizes Y, the A_1-factor in $C_G(x)$. Letting $D \leq N_j, D \cong Dih_{10}$ as above, we get that $C_G(D) \cong SO(5, \mathbb{C})^2$ and that, under one of the projections, the central involution z of Y maps to 1 or an involution conjugate to $\mathrm{diag}(-1, -1, -1, -1, 1)$ in $C_{F_i}(t) \cong SO(5, \mathbb{C})$ due to the invariant *symmetric* bilinear form. Thus, z is a product of evenly many $2A$ involutions as in (2.9) (the fundamental $SL(2, \mathbb{C})s$ come from $C(D') \cong 5A_4^2$) and so is in $2B$; however, the structure of $C_G(x)$ implies that it is in $2A$ since Y is a fundamental $SL(2, \mathbb{C})$. This contradiction proves that no N_i has an element of order 6.

We now prove that one of the N_i contains a copy of Alt_4. Since N_i is simple, it has no normal 2-complement, so by an old theorem of Frobenius, [Gor 1968] (7.4.5), there is a nonidentity 2-subgroup, Q, and an element u of odd order which normalizes but does not centralize Q. The possibilities here are $|u| = 3$ or 5. If 3, we are done, since $\langle u, t \rangle \cong \mathrm{Alt}_4$ for any involution $t \in Q$. So, we may assume that 3 does not occur this way for any i. The fact that N_i has no elements of order 10 means that u is fixed point free on Q. We may assume that Q is elementary abelian of order 16. Then, in the notation of the previous paragraph, every involution of Q is a product of involutions from the two factors F_i.

CASE 1. For each involution of Q, both components from the F_i are conjugate to either $\mathrm{diag}(-1, -1, 1, 1, 1)$ or $\mathrm{diag}(-1, -1, -1, -1, 1)$. In either case, every involution of Q is the product of central involutions from n pairwise commuting fundamental $SL(2, \mathbb{C})s$, where n is even and positive. Thus, involutions of Q are in $2B$, by (2.9). It follows from (3.8.ii) of [CoGr] that $C_G(Q)^\circ$ is a maximal torus and $C_G(Q)$ has component group 2^{1+6}. Since $C_G(Q)$ is solvable but contains N_j, for $j \neq i$, we have our contradiction.

CASE 2. Case 1 does not hold for either value of i. In either case, we may assume that the image of the natural map of Q to the F_i lies in the diagonal group, whose involutions are in $2A$ iff they are conjugate to $\mathrm{diag}(-1, -1, 1, 1, 1)$; see (2.9). Since $\langle u \rangle$ has three orbits on $Q^\#$, we deduce from knowing the three orbits of a 5-cycle permutation matrix on the diagonal group and from our being in Case 2 that exactly one orbit of $\langle u \rangle$ on Q consists of elements of $2B$. An inner product calculation with (2.7.iii) gives that $\dim C_G(\langle Q, u \rangle) = 4$. Thus, $C_G(\langle Q, u \rangle)$ is of type T_1^4 or A_1T_1. This forces N_j to be Alt_5, which contains an Alt_4 subgroup, and so we are done. ∎

3. **The proof.** Recall that L is a finite Lie primitive subgroup of G with socle $N = N_1 \times \cdots \times N_t$, a direct product of t nonabelian simple subgroups. In this section, we shall assume $t \geq 2$. From this, we derive that $N \cong \mathrm{Alt}_5 \times \mathrm{Alt}_6$, and describe $\chi|_N$. According to (2.10), $G = E_8(\mathbb{C})$ and there is an index, k, such that N_k contains a subgroup isomorphic to Alt_4.

LEMMA 3.1. *Let E be a four group in G all of whose involutions are conjugate. Set $Y = C_G(E)^{(\infty)}$. Then E is conjugate to a subgroup of T, Y is connected, and one of the following holds:*

(i) All involutions in Y are of type 2B, Y is of type D_4D_4 and $E \leq Z(Y)$.

(ii) All involutions in Y are of type 2A, Y is of type E_6 and $E \cap Y = 1$. Moreover, $C_G(Y)^{(\infty)}$ is a Lie subgroup of type A_2.

PROOF. See [CoGr 1987], (3.8) and (3.9). The statement about the centralizer of Y in (ii) follows from the fact that Y contains a conjugate of T. ∎

LEMMA 3.2. *Let S be a subgroup of G isomorphic to* Alt_4 *all of whose involutions have type 2B. Then $C_G(S)^{(\infty)}$ has type A_2A_2, A_2G_2, or G_2G_2 according as the trace of an order 3 element of S on $\mathbf{g}$ equals -4, 5, or* 14. *Moreover, $C_G\big(C_G(S)^{(\infty)}\big)^{(\infty)}$ is finite only in the first two cases, while in the last case, the centralizer is a subgroup of type A_1.*

PROOF. Let E be the four group in S. By Lemma 2.4, $C := C_G(E)^{(\infty)}$ is of type D_4D_4. It acts on $\mathbf{g}$ with character

$$(**) \qquad 8_*^{2-} \otimes 1_a + 1_a \otimes 8_*^{2-} + 8_* \otimes 8_* + 8_* \otimes 8_* + 8_* \otimes 8_*.$$

Choose an element $y \in S$ of order three. It induces an outer automorphism on C, which, by [CoGr 1987] is nontrivial on both factors D_4. By classical results on triality (*cf.* [Tits 1959]), the centralizer subgroup in each factor must then be of type A_2 or G_2, the centralizer of type A_2 acting irreducibly on each irreducible 8-dimensional module for D_4. Thus, $Y = C_G(S)$ is a closed subgroup of C of type A_2A_2, A_2G_2, or G_2G_2, as claimed. Moreover, the dimension of this subgroup is 16, 22, 28 in the respective cases and must equal

$$(1_a, \chi|_S) = \frac{1}{12}\big(248 + 3 \cdot (-8) + 8 \cdot \chi(y)\big).$$

Hence y has trace -4, 5, 14 in the respective cases.

On any 8-dimensional module for D_4, the triality subgroups of type A_2 and G_2 have restrictions 8_a and $1_a + 7_a$, respectively. On the Lie algebra for D_4, they have restrictions $8_a^{2-} = 8_a + 10_a + 10_b$ and $(1_a + 7_a)^{2-} = 2 \cdot 7_a + 14_a$, respectively. Straightforward character computations show that the trivial character occurs in $\chi|_Y$ only if Y has type G_2G_2 (coming from the triple $1_a \otimes 1_a$ part). Conversely, the centralizer subgroups of type G_2 have centralizer of type F_4. Thus, if Y has type G_2G_2, the centralizer F of one factor is isomorphic to $F_4(\mathbb{C})$ and contains the other factor, whence $C_G(Y) \geq C_F(Y)$, a subgroup of type A_1. ∎

LEMMA 3.3. *We have $t = 2$. Let $\{i,j\} = \{1,2\}$ and let E be a four subgroup of N_i all of whose involutions are G-conjugate. Then E is of the kind described in (i) of Lemma 3.1. Suppose furthermore that S is a subgroup of N_i isomorphic to* Alt_4. *Then N_j projects nontrivially into both factors of $Y = C_G(S)^{(\infty)}$ as in the previous lemma. In particular, N_j embeds in* $\mathrm{PSL}(3,\mathbb{C})$ *and so is isomorphic to one of* Alt_5, Alt_6.

PROOF. By definition of k, such an E is available in N_k, at least. Let $j \neq i$. If (ii) of Lemma 3.1 holds for some $E \leq N_i$, then, the subgroup $C_G\big(C_G(E)^{(\infty)}\big)^{(\infty)}$ is a group of type A_2 contradicting Corollary 2.4 above. Hence E is as described in (i) of Lemma 3.1. Again by Corollary 2.4, $C_G\big(C_G(S)^{(\infty)}\big)^{(\infty)}$ must be finite. By Lemma 3.2 this implies that Y has a factor of type A_2. The group N_j must project nontrivially on each factor of Y, for otherwise N_j lies in an $\mathrm{PSL}(3,\mathbb{C})$-subgroup of a D_4 factor which is irreducible on a natural 8-dimensional representation. The D_4-factor is isomorphic to $\mathrm{Spin}(8,\mathbb{C})$; its involutions form two conjugacy classes, one central (in the G-class 2B) and one noncentral (in the G-class 2A); it follows that the involutions of such N_j are of type 2A. In particular, N_j would have a four group as described by (ii) of Lemma 3.1, contradicting the first assertion of this lemma. Hence N_j embeds in both factors. Since at least one of them is of type A_2, the centralizer of N_1N_2 has trivial projection on at least one factor. Therefore, $t \leq 2$. Since Alt_5 and Alt_6 are the only simple $\{2,3,5\}$-subgroups of $\mathrm{PSL}(3,\mathbb{C})$ [Blich 1917], we need only reverse the roles of i and j to establish the lemma. ∎

LEMMA 3.4 (ELEMENTS OF ORDER 3 IN TRIALITY SUBGROUPS OF D_4D_4). *Let Y_1, Y_2 be triality subgroups of D_4 (of type A_2 or G_2) such that $C_G(S)^{(\infty)} = Y_1Y_2$, and suppose $y_i \in Y_i$ is an element of order 3 ($i = 1,2$) in Y_i lifting to an element of order 3 in the covering group of Y_i.*

(i) If Y_i has type A_2, y_i has trace -1 on an 8-dimensional module for D_4.

(ii) If Y_i has type G_2, y_i has trace -1 or 2 on an 8-dimensional module for D_4.

(iii) The product $y = y_1y_2$ satisfies $\chi(y) = 5$ if both y_i have trace -1 on the 8-dimensional D_4-modules and $\chi(y) = -4$ if one has trace -1 and the other trace 2 on the 8-dimensional D_4-modules.

PROOF. In Case (i), y_1 has trace 0 on the standard module 3_a for A_2 (as it has order 3 in the covering group) whence trace -1 on the adjoint module for A_2. In Case (ii) there are only two possibilities for y_1 up to conjugacy in $G_2(\mathbb{C})$, leading to trace -2 or 1 on the standard module 7_a for G_2 and hence trace -1 or 2 on a natural module 8_* for D_4. The lemma follows from use of these observations, the decomposition (**) of the adjoint module in the proof of (3.2). ∎

LEMMA 3.5. *If $N_1 \cong$* Alt_6, *then $N_2 \cong$* Alt_5 *and, up to automorphisms of $N = N_1N_2$,*

$$\chi|_N = 3_a \otimes 5_a + 3_b \otimes 5_b + 4_a \otimes (8_a + 8_b) + (3_a + 3_b) \otimes 9_a + 2 \cdot (5_a \otimes 10_a).$$

PROOF. First suppose $N_2 \cong \mathrm{Alt}_6$. Consider the group D of type D_4D_4 centralizing a subgroup of N_1 isomorphic to 2×2. Let $N_2 \leq X_1X_2$, where X_i is in the i-th factor of

D and $X_i \cong \mathrm{Alt}_6$ or $\mathrm{SL}(2,9)$. The fixed point subgroup of a triality automorphism on the i-th factor of D contains X_i. Therefore, $X_1 \cong X_2 \cong \mathrm{Alt}_6$. Consequently, the character of N_2 on the 8-dimensional modules for D may be identified with 8_a and 8_b for Alt_6. We use this to find $\chi|_{N_2}$ in terms of character values. We set $b_5 = \frac{-1+\sqrt{5}}{2}$ and write b_5^* for the algebraic conjugate $\frac{-1-\sqrt{5}}{2}$ so that

$$b_5 + b_5^* = -1, \quad b_5^2 = 1 - b_5, \quad b_5 b_5^* = -1.$$

Now, for elements of orders (1,2,3,3,4,5,5) the character values are:

$$8_a = (8, 0, -1, -1, 0, -b_5, -b_5^*)$$

and

$$8_b = (8, 0, -1, -1, 0, -b_5^*, -b_5)$$

Thus on the exterior square for 8_a:

$$8_a^{2-} = (28, -4, 1, 1, 0, -b_5, -b_5^*)$$

and on the tensor products

$$(64, 0, 1, 1, 0, 1 - b_5, 1 - b_5^*) \text{ in case } 8_a \otimes 8_a$$
$$(64, 0, 1, 1, 0, -1, -1) \text{ in case } 8_a \otimes 8_b$$

The full character on $\mathbf{g}$ is therefore

$$(248, -8, 5, 5, 0, 3 - 5b_5, 3 - 5b_5^*) \text{ in case } 8_a \otimes 8_a$$
$$(248, -8, 5, 5, 0, -2, -2) \text{ in case } 8_a \otimes 8_a.$$

An inner product computation shows

$$\dim C_{\mathbf{g}}(N_2) = \begin{cases} 3 & \text{in case } 8_a \otimes 8_a \\ 0 & \text{in case } 8_a \otimes 8_b \end{cases}.$$

If $\dim C_{\mathbf{g}}(N_2) > 0$, Lemma 2.2 gives that L must conjugate N_2 to N_1. But then in the case at hand, $N_1 \cong \mathrm{Alt}_6$ must act trivially on the 3-space $C_{\mathbf{g}}(N_2)$ (because there are no non-trivial 3-dimensional modules for Alt_6), whence $N_1 \times N_2$ centralizes $C_{\mathbf{g}}(N_2)$, contradicting Lemma 2.2. Consequently, the character of N_2 is $8_a \otimes 8_b$. Taking inner products with the irreducibles for Alt_6, we obtain

$$(*) \qquad \chi|_{N_2} = 3 \cdot (5_a + 5_b) + 4 \cdot (8_a + 8_b) + 6 \cdot 9_a + 10 \cdot 10_a.$$

Since Alt_6 does not have a 3-dimensional character without trivial constituents, use of (*) yields $N_1 \not\cong \mathrm{Alt}_6$.

Hence $N_1 \cong \mathrm{Alt}_5$. In particular, N_1 is normal in L, so by Lemma 2.3, $\chi|_{N_1}$ has no trivial constituents. According to [CoGr 1987] there is a unique character associated to fixed point free embedding of N_1 in $E_8(\mathbb{C})$; its character $\chi|_{N_1}$ is $14{\cdot}(3_a{+}3_b){+}16{\cdot}4_a{+}20{\cdot}5_a$. Apart from the character mentioned in the lemma there is only one other character compatible with both factors (*cf.* (*)):

$$\chi|_N = 3_a \otimes 5_b + 3_b \otimes 5_a + 4_a \otimes (8_a + 8_b) + (3_a + 3_b) \otimes 9_a + 2 \cdot (5_a \otimes 10_a).$$

(It helps to note that an irreducible for N_i of degree divisible by the order of a Sylow p-group of N_i vanishes on its p-singular elements, for $p = 3$ and 5). But this character is obtained from the one in the lemma by an automorphism of N induced by an automorphism of the abstract group Alt_6. ∎

LEMMA 3.6. *If* $N_2 \cong \mathrm{Alt}_5$, *then* $N_1 \cong \mathrm{Alt}_6$.

PROOF. If not, then by (3.3), $N_1 \cong \mathrm{Alt}_5$. We assume this and seek a contradiction.

We claim that the trace of an element of order 3 in each N_i is 5. Let $\{i,j\} = \{1,2\}$. Take a subgroup S of N_i, $S \cong \mathrm{Alt}_4$. Then $C := C_G(S)^{(\infty)}$ is of type A_2A_2 or A_2G_2 by Lemma 3.2. Let $y = y_1y_2$ be an element of order 3 in N_j, with y_1 in a factor of C of type A_2 and y_2 in the other factor. If C has type A_2A_2, then y has trace 5 on $\mathbf{g}$ by (2.5) and (3.2), while elements of order 3 in S have trace -4, so N_1 and N_2 are not conjugate. Moreover, each N_i is normal in L. Since Alt_5 has a unique fixed point free character on $\mathbf{g}$, at least one N_i has nonzero fixed points, a contradiction to (2.2). Therefore, C has type A_2G_2, and by Lemma 3.2 again, if $h \in S$ has order 3, $\chi(h) = 5$. Reversing the roles of N_i and N_j, we get $\chi(y) = 5$ whence the claim.

From (3.4), we deduce that both y_1 and y_2 have trace -1 on a natural module for a D_4 factor. The character table for Alt_5 shows that the restriction to N_j of a character 8_* for the D_4 factor must be of the form $3_* + 5_a$. But then N_j does not embed in a G_2-subgroup of D_4, contradicting $N_2 \leq C$ and (3.3). ∎

The conclusion is that L must have a normal subgroup N as described in Lemma 3.6. This establishes the first part of Theorem 1.1.

4. **Borovik's group.** In this section we prove the second part of Theorem 1.1, *i.e.*, we supply an existence proof of the Lie primitive group with socle $\mathrm{Alt}_5 \times \mathrm{Alt}_6$ and of its uniqueness up to conjugacy. It differs from Borovik's original approach in that he begins with a particular subgroup isomorphic to $\mathrm{PSL}(2, \mathbb{C})$ from Dynkin's list of subgroups of $E_8(\mathbb{C})$ [Dynk 1957] and takes an icosahedral subgroup of it. We begin with a subgroup $S \cong \mathrm{Alt}_4$ whose involutions are in class $2B$ and such that $C_G(S) \cong A_2(\mathbb{C})wr2$; see (3.4) and [CoGr 1987]. Let h be an element of order 3 in S. Since $\dim C_G(S) = 16$, we have $\chi(h) = -4$, $C_G(h) \cong 3A_8(\mathbb{C})$. Thus, the embedding of $C_G(S)$ in $C_G(h)$ is explained by identifying the 9-dimensional standard module for $C_G(h)$ with the tensor product of a pair of 3-dimensional spaces. Consequently, an involution of $C_G(S)$ not in either A_2-factor has eigenvalues $\{-1^4, 1^5\}$ on the 9-dimensional module, hence, by (2.9), is in G-class $2B$.

Up to conjugacy, there is a unique subgroup of $\mathrm{PSL}(3, \mathbb{C})$ isomorphic to Alt_6 (it is the image in $\mathrm{PSL}(3, \mathbb{C})$ of a subgroup $3^{\cdot}\,\mathrm{Alt}_6$ of $\mathrm{SL}(3, \mathbb{C})$ and is self-normalizing). Thus, in $C_G(S)$, there is up to conjugacy, a unique group of the form $\mathrm{Alt}_6\, wr2$ and this group contains one conjugacy class of subgroups isomorphic to Sym_6. This is the only way to get a Sym_6-subgroup of $C_G(S)$. By the preceeding paragraph, the involutions in the derived group of any such Sym_6-subgroup are in class $2B$.

We claim that if J is any Sym_5-subgroup of B, $C_{C_G(S)}(J) = 1$. We observe first that if Y is a subgroup of $C_G(S)^\circ$ such that $C_{C_G(S)^\circ}(Y) = 1$, then $C_{C_G(S)}(Y)$ has order at most 2. This remark applies to $Y = J'$. Since $N_{C_G(S)^\circ}(J') = J'$ and $N_{C_G(S)}(J')$ contains J, the claim follows.

Now, write B for a Sym_6-subgroup obtained as above. We study $C_G(B)$, which certainly contains S. The module $\mathbf{g}$ for $C_G(h)$ decomposes as $80_a + 9_a^{3-} + 9_b^{3-}$, where $80_a = 9_a \otimes 9_b - 1_a$ is the adjoint representation of $C_G(h)$. The embedding of B in $C_G(h)$ lifts to an action of B on the 9-dimensional module which, by the character table for Sym_6, is irreducible and which leaves invariant a nondegenerate symmetric bilinear form (the only other possible characters have degrees $(5, 1, 1, 1, 1)$, which would force the involutions of B' to be in class $2A$, a contradiction). Consequently, we may deduce the G-class of every element of B (straightforward with the above decomposition of $\mathbf{g}$ and the formula $\phi^{3-}(g) = [\phi(g)^3 - 3\phi(g)\phi(g^2) + 2\phi(g^3)]/6$ for the exterior cube of the character ϕ; on classes of cycle shapes $1, 2^2, 3, 3^2, 42, 5, 2, 2^3, 4, 6, 123$, the respective values under χ are $248, -8, 5, 5, 0, -2, 24, 24, 0, -3, -3$) and we may, because of the invariant bilinear form on the 9-dimensional module, arrange for an element $x \in C_G(B)$ to invert h under conjugation. Observe that $C_G(\langle h, B\rangle) = \langle h\rangle$. We get $C_G(B)$ finite either using this observation or by an inner product calculation with the traces given above. Define $U := \langle S, x\rangle$. By definition of S and x, $U' \geq S$. Note that U is finite since $U \leq C_G(B)$. We want to show that $C_G(B) = U \cong \mathrm{Alt}_5$.

Let J be a Sym_5-subgroup of B. On a 9-dimensional natural projective representation of $C_G(h)$, J has irreducibles of dimensions (4,5); also, $C_{C_G(h)}(J) \cong T_1$ and $C_{C_G(\langle h,x\rangle)}(J) \cong 2$. A straightforward inner product calculation with the above information shows that $\dim C_G(J) = 3$. Let F be a Frobenius group of order 20 in J. Since $C_G(F)$ is (by (2.7.iii)) isomorphic to $SO(5, \mathbb{C})$, the reductive subgroup $C_G(J)^\circ$ cannot be a rank three torus, so has type A_1. On the standard 5-dimensional module for $C_G(F)$, $C_G(J)^\circ$ has irreducibles of degrees 5, (1,1,3) or (2,2,1) since there is an invariant symmetric bilinear form. Only in Case (2,2,1) is $C_G(J)^\circ \cong SL(2, \mathbb{C})$, which contradicts an above statement that $C_{C_G(S)}(J) = 1$. Therefore, (2,2,1) does not occur and so $C_G(J) \cong PSL(2, \mathbb{C}) \times E$, where E is isomorphic to a finite subgroup of $O(2, \mathbb{C})$ via its action on the 0- or 2-dimensional fixed point space. Since $C_{C_G(h)}(J) \cong T_1$, the action of h on $C_G(J)$ fixes exactly a torus and h acts fixed point freely on E, whence $E \cong 2 \times 2$ or 1. We claim that $E = 1$. Suppose not. Then, the irreducibles for $C_G(J)$ have dimensions (1,1,3) and the action of h on E preserves its subgroup acting with determinant 1 on the 2-dimensional fixed point space of $C_G(J)$. This eliminates the possibility $E \cong 2 \times 2$ and so $E = 1$. So, $C_G(J) \cong PSL(2, \mathbb{C})$ (and $h \in C_G(J)$). The hypotheses on S and x and the classification of finite subgroups of $PSL(2, \mathbb{C})$ imply that $U \cong \mathrm{Alt}_5$ or Sym_4. If $U \cong \mathrm{Sym}_4$ then $U' = S$, $C_G(S) \cong A_2(\mathbb{C})wr2$ and either $C_G(U) \cong PSL(2, \mathbb{C})wr2$ (in case x normalizes the two A_2-factors) or $C_G(U) \cong PSL(3, \mathbb{C}) \times 2$ (in case x interchanges the two factors) and so $C_G(S)$ has no Sym_6-subgroup, a contradiction. Therefore, $U \cong \mathrm{Alt}_5$. Since $C_G(B)$ is a finite subgroup of $C_G(U)$ containing U, we conclude that $C_G(B) = U$.

To get the full normalizer of the finite semisimple group $N := U \times B$, we just recall the above remarks about $C_G(S)$ and $S \times B$ and use the fact that $N_G(S)$ has the shape $C_G(S)\langle S, r\rangle$, where r is an involution normalizing $C_G(S)$. We have $\langle S, r\rangle \cong \mathrm{Sym}_4$. A Frattini argument shows that r may be arranged to normalize B. Since the outer automorphism group of B has order 2 and $C_G(B) = U$, we have $N_G(N) = \langle r, U, B\rangle$ and $N_G(N)/U \cong \mathrm{Aut}(\mathrm{Alt}_6) \cong \mathrm{Alt}_6.2^2$. It follows from $\langle S, r\rangle \cong \mathrm{Sym}_4$ that $\langle U, r\rangle \cong \mathrm{Sym}_5$. We may choose r to be an involution which satisfies $C_B(r) \cong 5:4$. Since this is a subgroup of $C_G(S)$, it follows that r induces a graph automorphism on each A_2-factor of $C_G(S)$ (see remarks about the action of x in the previous paragraph).

We now verify Lie primitivity of N, which implies Lie primitivity for every subgroup between it and its normalizer. Suppose H is a closed Lie subgroup of G of positive dimension containing L. Then, we may assume that H is reductive and that N is Lie primitive in H. We prove $H = G$. If H° has a nontrivial central torus, N must act nontrivially on the connected center of H hence also on its Lie algebra, which has dimension at most 8. On the other hand, the character of Lemma 3.5 shows that the minimal dimension of a nonzero N-submodule of $\mathbf{g}$ is 15, a contradiction. Hence, H° is semisimple.

We argue that N must be in H°. For otherwise, on the set of components there is a nontrivial orbit $\{H_i \mid i \in I\}$, $5 \leq |I| \leq 8$. Every such H_i must have rank just 1 and, since the 2-rank of $E_8(\mathbb{C})$ is 9 (*cf.* [Adams 1986], [CoSe 1987] or [Gr 1991]), each must be an $\mathrm{SL}(2,\mathbb{C})$. Since the minimal degree of a faithful permutation of N is 11, one of the factors, say N_j, operates faithfully as inner automorphisms on $H^* := \langle H_k \mid k \in I\rangle$, whence $N_j \cong \mathrm{Alt}_5$ and so, if $\{i,j\} = \{1,2\}$, $N_i \cong \mathrm{Alt}_6$ and $|I| = 6$. Since the actions of N_i and N_j on H^* commute, N_i centralizes a diagonal subgroup of H^* isomorphic to $\mathrm{SL}(2,\mathbb{C})$ or $\mathrm{PSL}(2,\mathbb{C})$, contradicting fixed point freeness of N_i. Therefore, $N \leq H^\circ$.

We now have that N projects faithfully into each quasisimple factor of H, by fixed point freeness. By Lie primitivity of N in H, these projections are Lie primitive in the respective factors, which, by (2.8) are all $E_8(\mathbb{C})$. Therefore, $H = G$ and we are done.

5. **Remarks on isotypical alternating subgroups.** If L is a subgroup of G containing a normal subgroup $N_1 \cdots N_t$ whose factors are nonabelian simple subgroups which are L-conjugate, there exist a nonabelian finite simple group N_0 and group isomorphisms $\phi_i\colon N_0 \longrightarrow N_i$ such that $\phi_j\phi_i^{-1}\colon N_i \longrightarrow N_j$ coincides with the restriction to N_i of conjugation by an element of L for each $i,j \in \{1,\ldots,t\}$. In particular, if χ is a character of G, then $\chi \circ \phi_i = \chi \circ \phi_j$ for all i,j $(1 \leq i,j \leq t)$. We say that a subgroup M of G is *t-isotypical* if there is a subgroup M_0 of M and an isomorphism $\phi = (\phi_i)_{1\leq i \leq t}\colon M_0 \times M_0 \times \cdots \times M_0 \longrightarrow M$ such that $\chi \circ \phi_i = \chi \circ \phi_j$ for all i,j $(1 \leq i,j, \leq t)$, where χ is the adjoint character for E_8.

One might try to prove Theorem 1.1 via determination of characters of t-isotypical subgroups for $t > 1$, using feasible characters of simple subgroups [CoGr 1987] and [CoWa 1989] and Lemma 2.2.

For E_8 and $N_1 \cong \mathrm{Alt}_5$, so many 2-isotypical characters (with zero fixed points in $\mathbf{g}$) exist that this does not seem an efficient method.

The group Alt_6 has very few fixed-point-free 2-isotypical representations in $E_8(\mathbb{C})$: up to outer automorphisms and permutations of the factors, there are two:

$$1_a \otimes 8_a + 8_a \otimes 1_a + 2 \cdot 1_a \otimes 10_a + 2 \cdot 10_a \otimes 1_a + 3 \cdot 8_a \otimes 8_a$$

and

$$1_a \otimes 5_a + 5_a \otimes 1_a + 1_a \otimes 9_a + 9_a \otimes 1_a + 1_a \otimes 10_a + 10_a \otimes 1_a + 4 \cdot 5_a \otimes 5_a + 10_a \otimes 10_a.$$

In the respective cases, the fixed point space of N_1 in $\mathbf{g}$ has dimension 28 and 24. They lead to embeddings of N in D_4D_4 and A_4A_4. The character table of Alt_7 then rules out 2-isotypical representations of Alt_i for $i \geq 7$.

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